An Improved Window Based On Cosine Hyperbolic Function

M. Nouri, S. Sajjadi Ghaemmaghami, A. Falahati

Abstract—**A new simple form window with the application of FIR filter design based on the exponential function is proposed in this article. An improved window having a closed simple formula which is symmetric ameliorates ripple ratio in comparison with cosine hyperbolic window. The proposed window has been derived in the same way as Kaiser Window, but its advantages have no power series expansion in its time domain representation. Simulation results show that proposed window provides better ripple ratio characteristics which are so important for some applications. A comparison with Kaiser window shows that the proposed window reduces ripple ratio in about 6.4dB which is more than Kaiser's in the same mainlobe width. Moreover in comparison to cosine hyperbolic window, the proposed window decreases ripple ratio in about 6.5dB which is more than cosine hyperbolic's. The proposed window can realize different criteria of optimization and has lower cost of computation than its competitors.**

Index Terms— **Window functions; Kaiser Window; FIR filter design; Cosine hyperbolic window**

I. INTRODUCTION

FIR filters are particularly useful for applications where exact linear phase response is required. The FIR filter is generally implemented in a non-recursive way which guarantees a stable filter. FIR filter design essentially consists of two parts, approximation problem and realization problem. The approximation stage takes the specification and gives a transfer function through four steps [1,2]. They are as follows:

1) A desired or ideal response is chosen, usually in the frequency domain.
2) An allowed class of filters is chosen (e.g. the length N for a FIR filters).
3) A measure of the quality of approximation is chosen.
4) A method or algorithm is selected to find the best filter transfer function.

The realization part deals with choosing the structure to

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implement the transfer function which may be in the form of circuit diagram or in the form of a program. The essentially three well-known methods for FIR filter design are the window method, the frequency sampling technique and Optimal filter design methods [2]. The basic idea behind the window is to choose a proper ideal frequency-selective filter which always have a noncausal, infinite-duration impulse response and then truncate (or window) its impulse response $h_d[n]$ to obtain a linear-phase and causal FIR filter [3].

$$h[n]=h_d[n]\,w[n]\;;\quad w[n]=\begin{cases} f(n) & 0 \le n \le M \\ 0 & otherwise \end{cases} \quad (1)$$

Where $f(n)$ is function of n , (M+1) is the length, $h[n]$ represented as the product of the desired response $h_d[n]$ and a finite-duration "window", $w[n]$. So the Fourier transform of $h[n]$, $H(e^{jw})$, is the periodic convolution of the desired frequency response, $H_d\ (e^{jw})$, with Fourier transform of the window, $W(e^{jw})$. Thus, $H(e^{jw})$ will be a spread version of $H_d\ (e^{jw})$. Fourier transforms of windows can be expressed as sum of frequency-shifted Fourier transforms of the rectangular windows. Two desirable specifications for a window function are smaller main lobe width and good side lobe rejection (smaller ripple ratio). However these two requirements are incongruous, since for a given length, a window with a narrow main lobe has a poor side lobe rejection and contrariwise. The rectangular window has the narrowest mainlobe, it yields the sharpest transition of $H(e^{jw})$ at a discontinuity of $H_d\ (e^{jw})$. So by tapering the window effortlessly to zero, side lobes are greatly reduced in amplitude [2]. By increasing M, $W(e^{jw})$ becomes narrower, and the smoothing provided by $W(e^{jw})$ is reduced. The large sidelobes of $W(e^{jw})$ result in some undesirable ringing effects in the FIR frequency response $H(e^{jw})$, and also in relatively larger sidelobes in $H(e^{jw})$. So using windows that don't contain abrupt discontinuities in their time-domain characteristics, and have correspondingly low sidelobes in their frequency-domain characteristics is required [3].

There are different kind of windows and the best one is depending on the required application, Windows can be categorized as fixed or adjustable [9]. Fixed windows have only one independent parameter, namely, the window length which controls the main-lobe width. Adjustable windows have two or more independent parameters, namely, the window length, as in fixed windows, and one or more additional parameters that can control other window's characteristics.

The Kaiser window is a kind of two parameter windows, that have maximum energy concentration in the mainlobe, it control the mainlobe width and ripple ratio [4,8,9]. In this paper an improved two parameter window based on the exponential function is proposed, that performs better ripple ratio and lower sildelobe (6.42 db) compared to the Kaiser and Cosine hyperbolic windows, while having equal mainlobe width. Also its computation reduced because of having no power series. The paper is organized as follows: Section II presents the characterization of window to distinguish the windows performance, and introduces Cosh and Kaiser Windows. Section III introduces the proposed window and presents numerical simulations and discusses the final results. Section IV shows the time required to compute the window coefficients for the Cosh, Kaiser and proposed windows. Section V is given a numerical comparison example for the filters using the Proposed and Kaiser windows. Finally, conclusion is given in section VI after which the paper is equipped with related references.

II. CHARACTERIZATION OF WINDOW

A window, $w(nT)$, with a length of N is a time domain function which is defined by:

$$w(nT) = \begin{cases} nonzero & |n| \le (N-1)/2 \\ 0 & other \end{cases} \tag{2}$$

Windows are generally compared and classified in terms of their spectral characteristics. The frequency spectrum of $w(nT)$ can be introduced as [7]:

$$W(e^{jwT}) = W_0(e^{jwT})e^{-\frac{jw(N-1)T}{2}} \tag{3}$$

Where $W(e^{jwT})$ is called the amplitude function, N is the window length, and T is the space of time between samples. Two parameters of windows in general are the null-to null width B_N and the main-lobe width B_R. These quantities are defined as $B_N = 2\omega_N$ and $B_R = 2\omega_R$, where ω_N and ω_R are the half null-to-null and half mainlobe widths, respectively, as
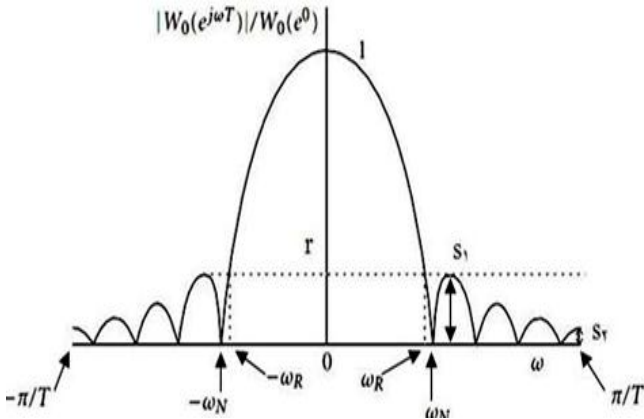

Figure 1. A typical window's normalized amplitude spectrum

shown in Fig. 1, an important window parameter is the ripple ratio r which is defined as

$$r = \frac{maximum\ side-lobe\ amplitude}{mainlobe\ amplitude} \tag{4}$$

Having small proportion less than unity permit to work with the bilateral of r in dB, that is

$$R = 20 \log\left(\frac{1}{r}\right) \tag{5}$$

R clarifies as the minimum side-lobe attenuation relative to the main lobe and $-R$ is the ripple ratio in dB. S is the side-lobe roll-off ratio, which is defined as

$$S = \frac{s_1}{s_2} \tag{6}$$

s_1 is the largest side lobe and s_2 is the lower one which is furthest from the main lobe. If S is the side-lobe roll-off ratio in dB, then s is given by

$$s = 10^{\frac{S}{20}} \tag{7}$$

These spectral characteristics are important performance measures for windows.

A. Kaiser window

Kaiser window is one of the most useful and optimum windows. It is optimum in the sense of providing a large mainlobe width for a given stopband attenuation, which implies the sharpest transition width [1]. The trade-off between the mainlobe width and sidelobe area is quantified by seeking the window function that is maximally concentrated around w=0 in the frequency domain [2].

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2 \tag{8}$$

In discrete time domain, Kaiser Window is defined by [5]:

$$w_k(n) = \begin{cases} \frac{I_0\left(\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\alpha)} & |n| \le \frac{N-1}{2} \\ 0 & other \end{cases} \tag{9}$$

Where α is the shape parameter, N is the length of window and I0(x) is the modified Bessel function of the first kind of order zero.

B. Cosh Window

The hyperbolic cosine of x is expressed as:

$$cosh(x) = \frac{e^x + e^{-x}}{2} \tag{10}$$

$$cosh(x) = \sum_{i=0}^{+\infty} \frac{x^{2n}}{(2n)!} \tag{11}$$

Fig. 2, shows that the functions Cosine hyperbolic(x) and $I_0(x)$ have the same Fourier series characteristics [7]. Cosine-

hyperbolic window is proposed as:

$$w_k(n) = \begin{cases} \dfrac{Cosh(\alpha\sqrt{1-\left(\dfrac{2n}{N-1}\right)^2}}{Cosh(\alpha)} & |n| \leq \dfrac{N-1}{2} \\ 0 & other \end{cases} \quad (11)$$

This window provides better sidelobe roll-off ratio, but worse ripple ratio for the same window length and mainlobe width compared with Kaiser Window. It has the advantage of having no power series expansion in its time domain function so the Cosine hyperbolic window has less computation compared with Kaiser one.

III. PROPOSED WINDOW

This new window is based on Cosine hyperbolic window that is optimized by applying a cost function for diminishing ripple ratio. The result of this optimization is changing the power of exponential phrases and making a new α function, the adjustable shape parameter, these changes can be shown as:

$$Cosh(x) = \frac{e^{x^{1.2}} + e^{-x^{1.2}}}{2} \quad (12)$$

From Fig. 2, Cosine hyperbolic($x^{1.2}$), $I_0(x)$ and Cosine hyperbolic(x) have similar Fourier series characteristics. Fig. 3 shows the frequency domain plots of Cosine hyperbolic window for various α values with N=51. From this figure, it is easily observed that as α increases the mainlobe width and the ripple ratio become wider and smaller, respectively. For some applications such as the spectrum analysis, the design equations which define the window parameters in terms of the spectrum parameters are required [3]. From Fig. 4, an approximate relationship for α in terms of R can be found by using the quadratic polynomial curve fitting method as:
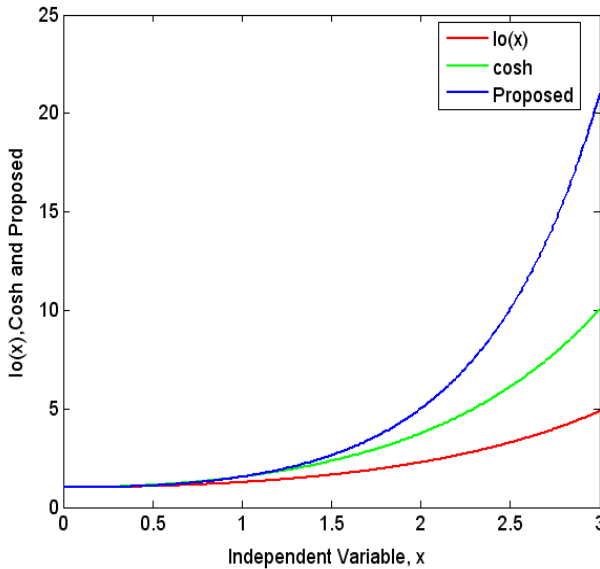
$$\alpha = \begin{cases} 0, & R > -13.26 \\ -3.936 \times 10^{-5}R^3 \\ -0.0046665 \times R^2 \\ -0.30504 \times R - 2.6125, & -50 < R \leq -13.26 \\ 6.1545 \times 10^{-5}R^2 \\ -0.1302 \times R \\ -0.74597, & -120 < R \leq -50 \end{cases}$$

(14)

In Fig. 4, the approximation model for the adjustable shape parameter given by Eq.(14) is plotted. It is seen that the proposed window has better performance than Kaiser window, also comparing with Cosine hyperbolic window shows that the ripple ratio is almost lower, and at R= -25.5 the Cosine hyperbolic window replace it and becomes lower than proposed one.

The second design equation is the relation between the window's length and the ripple ratio. To predict the Window's length for a given quantities R and W_R, the normalized width D=2W_R(N-1) is used [5]. The relation between D and R for Cosine hyperbolic window is given in Eq.(15). By using quadratic polynomial curve fitting method for figure 5, an approximate design relationship between D and R can be established as:

$$D = \begin{cases} 0, & R > -13.26 \\ -0.0028658R^2 \\ -1.555 \times R & -50 < R \leq -13.26 \\ -1.2875, \\ -8.702 \times 10^{-4}R^2 \\ -0.6184R & -120 < R \leq -50 \\ +1.972, \end{cases}$$

(15)

The approximation model for the normalized width given by Eq.(15) is plotted in Fig 5. These numerical results are summarized in Table I. It is seen that the innovation gradient of the proposed window is faster and better than the Cosine hyperbolic window.
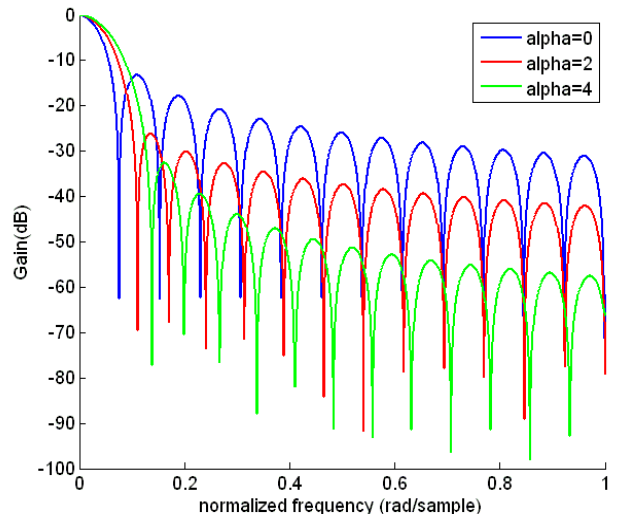

Figure 2. Comparison of the functions $I_0(x)$, Cosine hyperbolic(x) and proposed window

Figure 3. Proposed window spectrum in dB for $\alpha = 0, 2, 4$ and N=50
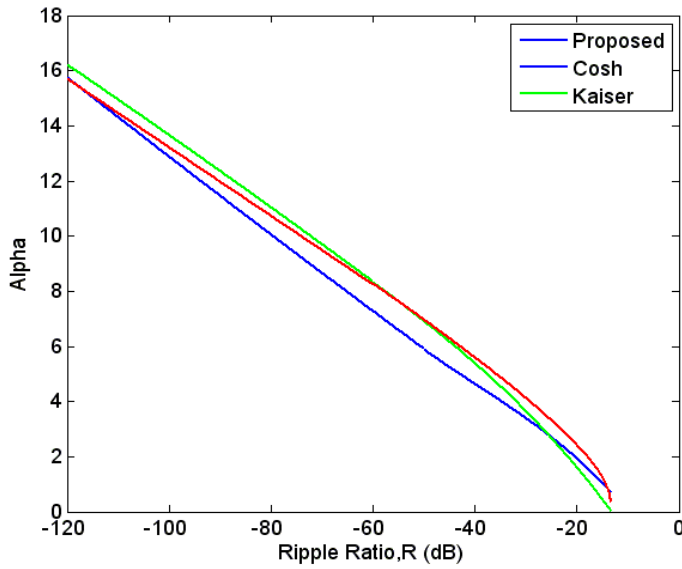
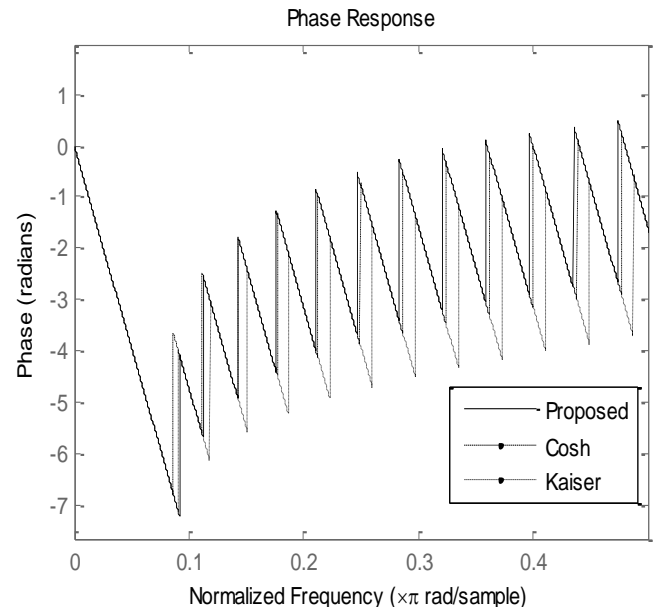Figure 4. The relation between α and R for the proposed window

Figure 6. Phase comparison between the Cosine hyperbolic, Kaiser and proposed windows for $N = 50$

TABLE I
COMPARISON BETWEEN THE COSINE HYPERBOLIC AND PROPOSED WINDOW

Normalized width (D_w)	10	25	40	55	70
Proposed Ripple ratio (R)	-17.12	-42.4	-71.4	-95.48	-124.2
Cos hyperbolic Ripple ratio (R)	-18.02	-38	-64.7	-87.2	-110.5

A. Comparison with Kaiser Window

The comparison of the proposed and Kaiser Windows in terms of the normalized frequency for N=50 is plotted in Fig .7, and the corresponding data is given in Table II. Fig .7 shows that the side lobe peak of the proposed window is 6.43 *dB* beneath the Kaiser window with the same mainlobe width for N = 50. The ripple ratio is -50.43 for the proposed window, but this quantity is equal with -44 for the Kaiser one.

Sidelobe roll-off ratio for the proposed window has the quantity of 14, as this parameter is 32 for Cosine hyperbolic window and it is equal with 21.5 for the Kaiser window.

Fig. 6 plotted the phase response between the Cosine hyperbolic, Kaiser and proposed windows.
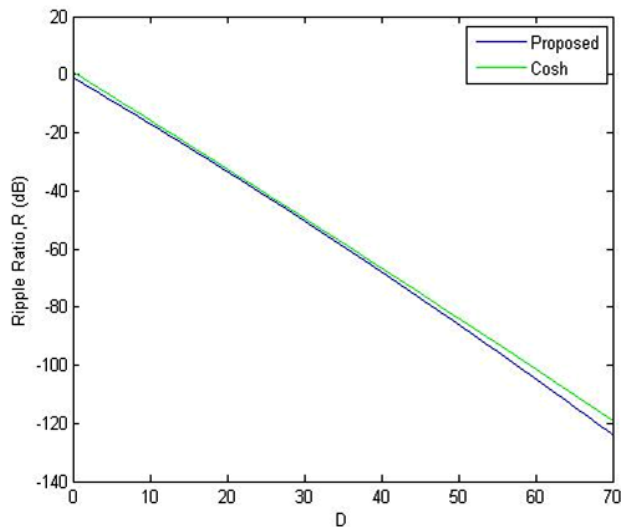

Figure 5. Relation between ripple ratio and D for cosine hyperbolic and proposed window in $N = 50$
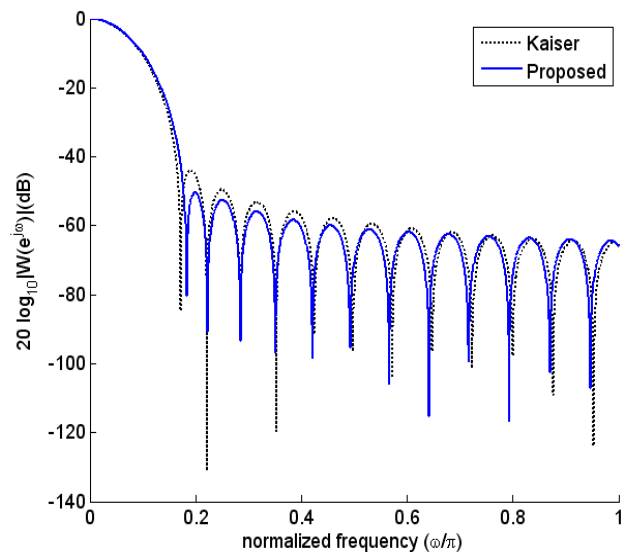

Figure 7. Comparison between the proposed and Kaiser windows with N=50

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TABLE II
DATA FOR THE KAISER AND PROPOSED WINDOWS SPECTRUM WITH N=50

	W_r	R	S
proposed window	0.169	-44.01	14
Kaiser window	0.169	-50.43	21.5

B. Comparison with Cosine hyperbolic Window

The comparison with Cosine hyperbolic window for N=50 are given below. The simulation result is shown in Fig. 8. The proposed window offers a reduced ripple ratio than the Cosine hyperbolic window. The Cosine hyperbolic window gives a smaller ripple ratio and the corresponding data is given in Table II. They show that the sidelobe peak of the proposed window is 6.53 *dB* beneath the Cosine hyperbolic window with the same mainlobe width for N is 50. And the ripple ratio is -50.43 for the proposed window, but this quantity is -43.89 for the Cosine hyperbolic one.

IV. COMPUTATIONAL TIME

From Fig. 9 shows the time required to compute the window coefficients for the Cosine hyperbolic, Kaiser and the proposed windows. From Fig. 9, it can be easily seen that the elapsed time for the Cosine hyperbolic window changes from 0.004 to 0.048 ms, while it changes from 0.14 to 0.59 ms for the Kaiser window, and it changes from 0.041 to 0.09 ms for

TABLE II
DATA FOR THE COSINE HYPERBOLIC AND PROPOSED WINDOWS SPECTRUM WITH N=50

	W_r	R	S
proposed window	0.174	-50.43	14
Cosine hyperbolic window	0.174	-43.89	32

Figure 9. Computation time comparison between the proposed,Cosine hyperbolic and Kaiser windows for various window length

the proposed window. So it is obvious from this figure, the Cosine hyperbolic window is computationally efficient compared to the Kaiser window due to Having no power series expansion in its time domain representation and even though the proposed window is computationally better compared to the Kaiser window but it is a bit computationally compared with Cosine hyperbolic window.

V. APPLICATION TO FIR FILTER DESIGN

FIR filter design is almost entirely restricted to discrete time implementations. The design techniques for FIR filters are based on directly approximating the desired frequency response of the discrete time system [2]. In order to show the efficiency of the proposed window and compare the results with the other windows, an example of designing an FIR low pass filter by windowing of an ideal IIR low pass filter is considered. Having a cut-off frequency of ω_C, the impulse
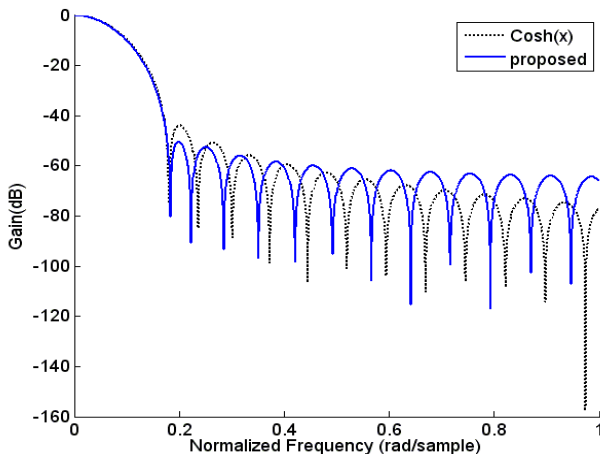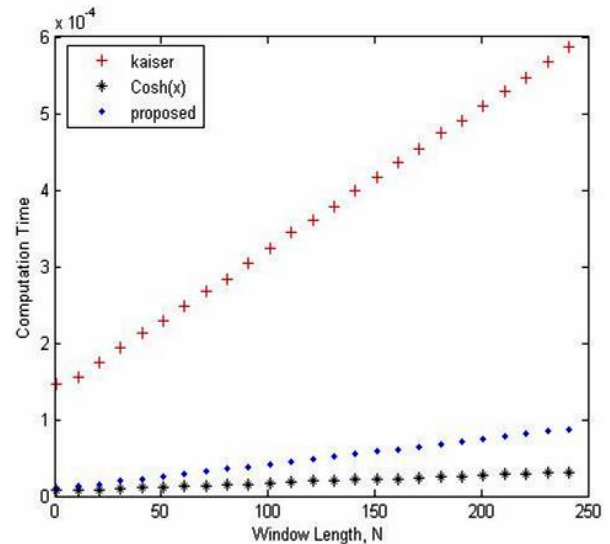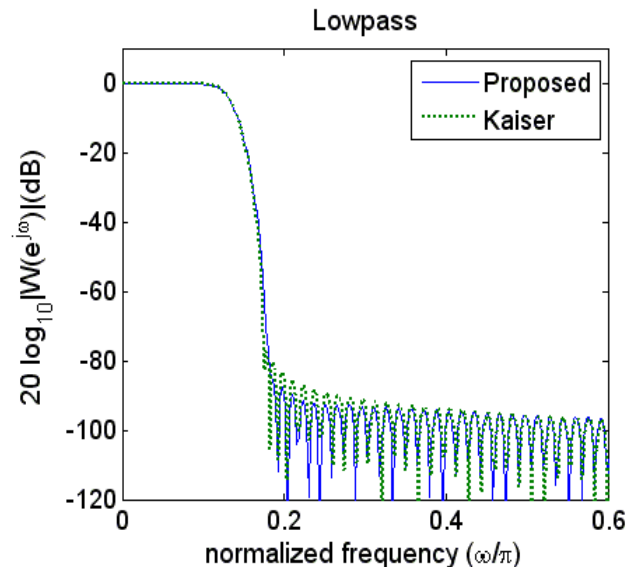

Figure 8. Comparison between the proposed and Cosine hyperbolic windows with N=50 and $\beta = 6$

Figure 9. The filters designed by the Proposed and Kaiser Windows for $w_{ct} = 0.4\,\pi$ and $\Delta w = 0.2$ rad /sample with $N = 50$

response of an ideal low pass filter is:

$$h_{nc}(nT) = w(nT)h_{id}(nT) \qquad (16)$$

$$h_{id}(nT) = \frac{Sin(w_c n)}{\pi n} \qquad (17)$$

VI. CONCLUSION

In this paper an improved class of window family based on cosine hyperbolic function is proposed. The proposed window has been derived in the same way of the derivation of Kaiser Window, but it has the advantage of having no power series expansion in its time domain function. The spectrum comparisons with Kaiser Window for the same window length and normalized width show that window provides better ripple ratio characteristics which so imperative for some applications. The last spectrum comparison is performed with window, and two specific examples show signs of that for narrower mainlobe width and smaller ripple ratio. Powered Cosine hyperbolic window's disadvantages are having much more computation than the Cosine hyperbolic window, although this window again has less computation compared with the Kaiser one.

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